17.6.2021: A teraz z ineho sudka: fixing the mistake from the Project of Disseration

I think it's time to fix the mistake from my Project of Dissertation. The problem is the Theorem 4.1.2 (page 41), which explains under what conditions is the correcting term l ($I_O(F,G) = mn + t + l$) equal to 0. This theorem gives some conditions for every common tangent of the intersecting curves F and G. These are correct if the tangent is of higher multiplicity on one curve than on the other. If both of the curves have this tangent with the same multiplicity, the statement of this theorem is incorrect. So let's fix this. (i'm going to use a little bit different notation)

Theorem 0.1. Let F and G be curves defined by polynomials

$$F = F_m + F_{m+1} + \cdots, \quad (1)$$
$$G = G_n + G_{n+1} + \cdots, \quad (2)$$

such that F and G have t tangents in common at 0. Then

$$I_O(F,G) = mn + t, \quad (3)$$

(i.e. the correcting term $l = I_O(F,G) - mn - t$ is equal to zero) if an only if the following condition is satisfied for each common tangent L of F and G at O (of multiplicity r and s respectively):

- If $r > s$, then F_{m+1} is not divisible by L.
- If $r < s$, then G_{n+1} is not divisible by L.
- If $r = s$, then $v_0 a_s \neq b_0 u_s$, where v_0, u_s, b_0, a_s ($a_s, u_s \neq 0$) are the coefficients of F and G after the transformation which maps L onto y. After this transformation, the polynomials are

$$\begin{aligned} F &= [F_m] + [F_{m+1}] + \cdots = [a_s x^{m-s} y^s + \cdots + a_m y^m] + [b_0 x^{m+1} + \cdots + b_{m+1} y^{m+1}] + \cdots \\ G &= [G_n] + [G_{n+1}] + \cdots = [u_s x^{n-s} y^s + \cdots + u_n y^n] + [v_0 x^{n+1} + \cdots + v_{n+1} y^{n+1}] + \cdots \end{aligned} \quad (4)$$

  - **REMARK:** F_{m+1} is divisible by L iff $b_0 = 0$ and G_{n+1} is divisible by L iff $v_0 = 0$. Therefore
    * if both F_{m+1} and G_{n+1} are divisible by L, the condition $v_0 a_s \neq b_0 u_s$ never holds.
    * If exactly one of the polynomials F_{m+1}, G_{n+1} is divisible by L and the other is not, the condition always holds.
    * If both F_{m+1}, G_{n+1} are not divisible by L, then we need to check the numbers values v_0, u_s, b_0, a_s.

Proof. Proof of the $r > s$ and $r < s$ case is in the Project of Dissertation. Same applies to the case $r = s$, where at least one of the polynomials F_{m+1} and G_{n+1} is divisible by L.
The rest is by brute force. If both are not divisible by L, then after splitting the curves into branches at O, both would get exactly one branch with the tangent L. This branch has a parametrization

$$B = \left(t^s, t^{s+1}\left(\sqrt[s]{\frac{-v_0}{u_s}} + \alpha_1 t + \alpha_2 t^2 + \cdots\right)\right)$$

(the values of α_i are not important, they can be whatever)

for the curve G (and analogous for the curve F). After substituting into the polynomial F, we get the result. □

Remark. With the condition $v_0 a_s \neq b_0 u_s$, we have returned to the exponent of contact of two branches. We are basically asking, if the two branches do have the same beginning of the expansion. Therefore, what the theorem says is, that in some cases there is easier way of checking if $I_O(F,G) = mn + t$, but in this last case, we need to actually check the expansion. But it's a little pre-computed. Is this good for anything? I don't know.