14.7.2021: Some more notes for the case of the difference between Orechovnik and the Blowup.

The section of the date 17.5.2021 contains this:

EXAMPLE 2: intersections with identical blowup sequences, but different ꩜ sequences

Let

$$\begin{aligned} F_1 &= x^2 - y^5 && (1) \\ G_1 &= x^4 - y^7 && (2) \end{aligned}$$

- **BLOWUP:** $I_O(F_1, G_1) = 2 \cdot 4 + 2 \cdot 3 = 8 + 6 = 14$
- ꩜: $I_O(F_1, G_1) = 1 + 2 + 2 + 2 + 2 + 2 + 2 + 1 = 14$

$$\begin{aligned} F_2 &= x^2 - y^5 && (3) \\ G_2 &= x^4 + xy^5 - y^7 && (4) \end{aligned}$$

- **BLOWUP:** $I_O(F_2, G_2) = 2 \cdot 4 + 2 \cdot 3 = 8 + 6 = 14$
- ꩜: $I_O(F_2, G_2) = 1 + 2 + 2 + 2 + 2 + 2 + 1 + 1 + 1 = 14$

The question is what is the difference between these two intersections that ꩜ sees.
The last curve in this example ($G_2 = x^4 - y^7 + xy^5$) is the only one of all these curves which splits into distinct branches. All the curves of type $y^a = x^b$ can be parametrized by $B(t) : (t^b, t^a)$. The branches of G_2 are:

$$\begin{aligned} C_1(t) &: \left(t^2, t + \frac{1}{2}t^2 + \cdots\right) \\ C_2(t) &: \left(t^5, -t^3 - \frac{1}{5}t^4 + \cdots\right) \end{aligned} \qquad (5)$$

When intersecting the individual branches C_1 and C_2 with the curve F_2, the intersection multiplicity splits into

$$\begin{aligned} I_O(F_2, G_2) &= I_O(F_2, C_1) + I_O(F_2, C_2), \\ 14 &= 4 + 10. \end{aligned} \qquad (6)$$

This is because

$$\begin{aligned} F_2(C_1(t)) &= t^4 - \left(t^2, t + \frac{1}{2}t^2 + \cdots\right)^5 = t^4 + \text{ (terms of higher degree)}, \\ F_2(C_2(t)) &= t^{10} - \left(t^5, -t^3 - \frac{1}{5}t^4 + \cdots\right)^5 = t^{10} + \text{ (terms of higher degree)}. \end{aligned} \qquad (7)$$

Maybe we are forgetting the most obvious interpretation of the ꩜ sequence, which are the dimensions of the $\mathcal{O}/(I^k, F, G)$ vector spaces. (it's more like a definition than a interpretation).
The sequence ꩜$_{F_2,G_2}$: 1, 2, 2, 2, 2, 2, 1, 1, 1 means that

$$\begin{aligned} \dim(\mathcal{O}/(I^0, F, G)) &= 0, \\ \dim(\mathcal{O}/(I^1, F, G)) &= 1 \quad (= \dim(\mathcal{O}/(I^0, F, G)) + 1), \\ \dim(\mathcal{O}/(I^2, F, G)) &= 3 \quad (= \dim(\mathcal{O}/(I^1, F, G)) + 2), \\ \dim(\mathcal{O}/(I^3, F, G)) &= 5 \quad (= \dim(\mathcal{O}/(I^2, F, G)) + 2), \\ \dim(\mathcal{O}/(I^4, F, G)) &= 7 \quad \cdots, \\ \dim(\mathcal{O}/(I^5, F, G)) &= 9, \\ \dim(\mathcal{O}/(I^6, F, G)) &= 11, \\ \dim(\mathcal{O}/(I^7, F, G)) &= 12, \\ \dim(\mathcal{O}/(I^8, F, G)) &= 13, \\ \dim(\mathcal{O}/(I^9, F, G)) &= 14 = I_O(F, G) \\ \dim(\mathcal{O}/(I^{10}, F, G)) &= 14 \\ &\cdots \end{aligned} \qquad (8)$$