1 novinky

Let F and G be affine algebraic curves defined by polynomials

$$F = F_m + F_{m+1} + \cdots , \tag{1}$$
$$G = G_n + G_{n+1} + \cdots , \tag{2}$$

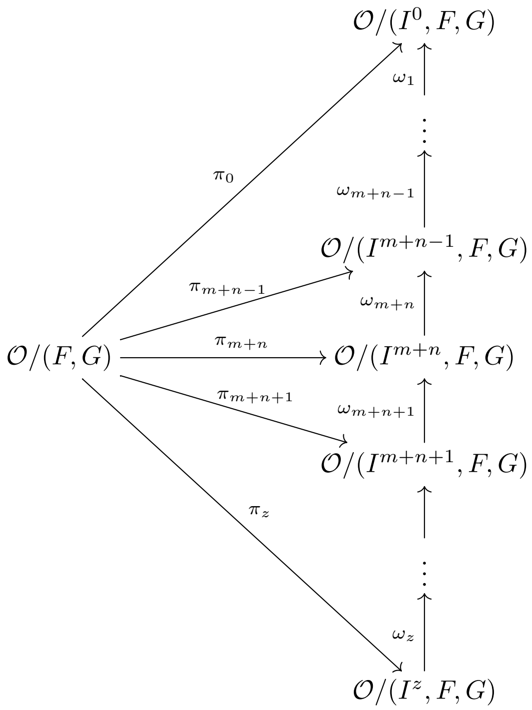
where $m, n > 0$, $F_m \neq 0$, $G_n \neq 0$. Let $\mathcal{O}$ be the local ring at O, and let I be the ideal (x, y). So far, we have used this beautiful diagram of vector spaces and linear maps from Fulton [W. Fulton: Algebraic Curves, p. 38]

$$\begin{array}{ccccccc}
k[x,y]/I^n \times k[x,y]/I^m & \xrightarrow{\psi} & k[x,y]/I^{m+n} & \xrightarrow{\varphi} & k[x,y]/(I^{m+n}, F, G) & \longrightarrow & 0 \\
 & & & & \Big\downarrow \alpha & & \\
 & & \mathcal{O}/(F,G) & \xrightarrow{\pi} & \mathcal{O}/(I^{m+n}, F, G) & \longrightarrow & 0,
\end{array}$$

where φ and π are natural surjective homomorphisms, α is isomorphism, and ψ is defined by $\psi(A, B) = AF - BG$. The top row forms an exact sequence. Thanks to this diagram, we know that the intersection multiplicity is equal to

$$I_O(F, G) = \dim(\mathcal{O}/(F, G)) = mn + \dim(\ker(\pi)) + \dim(\ker(\psi)) \tag{3}$$

Now we can create a similar diagram with more maps and spaces.



where $\mathcal{O}/(I^z, F, G) \cong \mathcal{O}/(F, G)$, which means that the map π_z is an isomorphism.

QUESTION I:
How do we know there is such z for each pair of curves? Can we find one for each pair of curves?

All the maps are surjective linear maps and $\dim(\mathcal{O}/(I^0, F, G)) = 0$, therefore

$$I_O(F, G) = \dim(\mathcal{O}/(F, G)) = \dim(\ker(\pi_0)) = \dim(\ker(\omega_z)) + \cdots + \dim(\ker(\omega_1)). \tag{4}$$

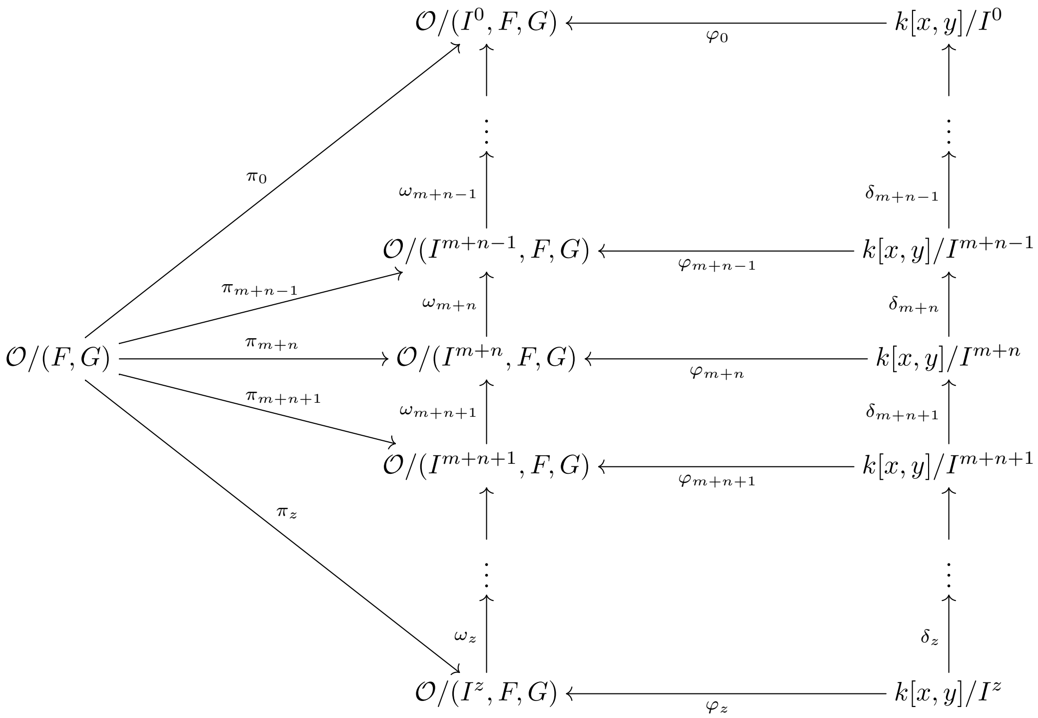
We have written the intersection multiplicity as a sum of integers which depend on the dimensions of the vector spaces $\mathcal{O}/(I^i, F, G)$.

QUESTION II:
Is there a meaning behind this? Possibly some geometry? Do the maps ω_i have some nice properties?

The maps maps ω_i describe the differences of dimensions of the vector spaces $\mathcal{O}/(I^i, F, G)$. We want to know more about them.

1.1 How to know more about ω_i

Now we can have add some more vector spaces and maps to our diagram.



(The original maps π and φ from the Fulton's diagram are now called π_{m+n} and φ_{m+n}.) All these maps are surjecive linear maps, and all the squares in this diagram are commutative. Therefore for each ω_i we know that

$$\dim(\ker(\omega_i)) + \dim(\ker(\varphi_i)) = \dim(\ker(\delta_i)) + \dim(\ker(\varphi_{i-1})). \quad (5)$$

This can be done also for bigger squares,

$$\dim(\ker(\omega_i)) + \cdots + \dim(\ker(\omega_{i-k})) + \dim(\ker(\varphi_i)) = \dim(\ker(\delta_i)) + \cdots + \dim(\ker(\delta_{i-k})) + \dim(\ker(\varphi_{i-k})). \quad (6)$$

If we do this for the biggest square of the diagram, we get

$$\sum_{i=1}^{z} \dim(\ker(\omega_i)) + \dim(\ker(\varphi_z)) = \sum_{i=1}^{z} \dim(\ker(\delta_i)) + \dim(\ker(\varphi_0)). \quad (7)$$

But $\sum_{i=1}^{z} \dim(\ker(\omega_i)) = \dim(\mathcal{O}/(I^z, F, G)) = \dim(\mathcal{O}/(F, G)) = I_O(F, G)$, and $\dim(\ker(\varphi_0)) = 0$, so we obtain a new formula for the intersection multiplicity,

$$I_O(F, G) = \sum_{i=1}^{z} \dim(\ker(\delta_i)) - \dim(\ker(\varphi_z)). \quad (8)$$

Since $\dim(\ker(\delta_i)) = i$ for any choice of F and G, the first part is boring. We need to focus on the map φ_i. This map can help us with the new intersection multiplicity formula, but also with individual maps ω_i.

1.2 How to know more about φ_i

To know more about φ_i, we can use Fulton's another trick, the map ψ. For each $i \geq \max\{m, n\}$ we have the following exact sequence

$$k[x,y]/I^{i-m} \times k[x,y]/I^{i-n} \xrightarrow{\psi_i} k[x,y]/I^i \xrightarrow{\varphi_i} \mathcal{O}/(I^i, F, G),$$

where $\psi_i(A, B) = AF - BG$. Now we can use the exactness in $k[x,y]/I^i$ to find $\ker(\varphi_i)$. We know that

$$\ker(\varphi_i) \cong \mathrm{Im}(\psi_i) \cong \left(k[x,y]/I^{i-m} \times k[x,y]/I^{i-n}\right) / \ker(\psi_i). \quad (9)$$

The dimension of the space $k[x,y]/I^{i-m} \times k[x,y]/I^{i-n}$ is pretty straightforward.

$$\dim\left(k[x,y]/I^{i-m} \times k[x,y]/I^{i-n}\right) = \frac{1}{2}((i-m+1)(i-m) + (i-n+1)(i-n)). \quad (10)$$

We have already spent some time on $\dim(\ker(\psi_{m+n}))$. And there are similarities between $\ker(\psi_i)$ and $\ker(\psi_j)$.

QUESTION III:
What is the relationship between $\ker(\psi_i)$ and $\ker(\psi_j)$ for various combinations of i and j?

Now we can substitute into (8).

$$\begin{aligned} I_O(F,G) &= \sum_{i=1}^{z} \dim(\ker(\delta_i)) - \dim(\ker(\varphi_z)) = \\ &= \sum_{i=1}^{z} \dim(\ker(\delta_i)) - \big(\dim(k[x,y]/I^{z-m} \times k[x,y]/I^{z-n}) - \dim(\ker(\psi_z))\big) = \\ &= \frac{1}{2}(1+z)z - \frac{1}{2}((z-m+1)(z-m) + (z-n+1)(z-n)) + \dim(\ker(\psi_z)) = \\ &= \frac{1}{2}(z^2 - (z-m)^2 - (z-n)^2 - z + m + n) + \dim(\ker(\psi_z)) = \\ &= z^2\left(-\frac{1}{2}\right) + z\left(m+n-\frac{1}{2}\right) + \frac{1}{2}(m+n-m^2-n^2) + \dim(\ker(\psi_z)) = \Xi_{m,n}(z) + \dim(\ker(\psi_z)). \end{aligned} \tag{11}$$

Which is a nice formula for intersection multiplicity. I've decided to write the first part as a polynomial in z, because that is how I feel it.

QUESTION IV:
We need to have a closer look on the polynomial $\Xi_{m,n}(z) = z^2\left(-\frac{1}{2}\right) + z\left(m+n-\frac{1}{2}\right) + \frac{1}{2}(m+n-m^2-n^2)$. Maybe its shape could help us with the search for the sufficient value of z. Or give us some kind of upper/lower bound for intersection multiplicity.

UPDATE 21.2.2021:
Under what conditions is $\Xi_{m,n}(z) = z^2\left(-\frac{1}{2}\right) + z\left(m+n-\frac{1}{2}\right) + \frac{1}{2}(m+n-m^2-n^2)$ an integer? Because it needs to be an integer.

UPDATE2 21.2.2021:
Ok, nevermind, it is an integer for all m, n, z integers
$\Xi_{m,n}(z) = z^2\left(-\frac{1}{2}\right) + z\left(m+n-\frac{1}{2}\right) + \frac{1}{2}(m+n-m^2-n^2) = z(m+n) - \frac{1}{2}[z(z+1) - m(m-1) - n(n-1)]$
$= (\text{integer}) - \frac{1}{2}[(\text{even number})]$

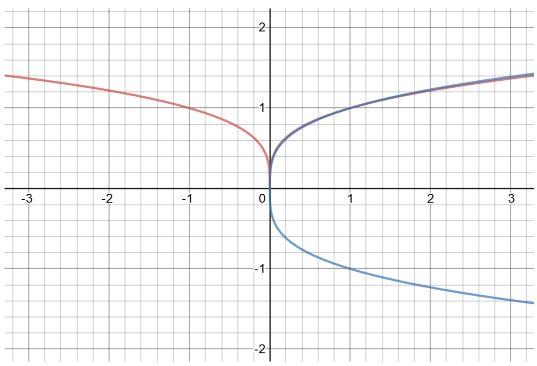
UPDATE3 21.2.2021:
Could we make some kind of algorithm for $\dim(\ker(\psi_z))$? Because that would be nice.

UPDATE4 21.2.2021:
Maybe if the number z_0 (the lowest z, such that $\mathcal{O}/(F,G) = \mathcal{O}/(I^z, F, G)$) is somewhere where $\Xi_{m,n}(z)$ is still positive, we could make some claims about the intersection multiplicity. But I don't know if this is a good idea.

2 example

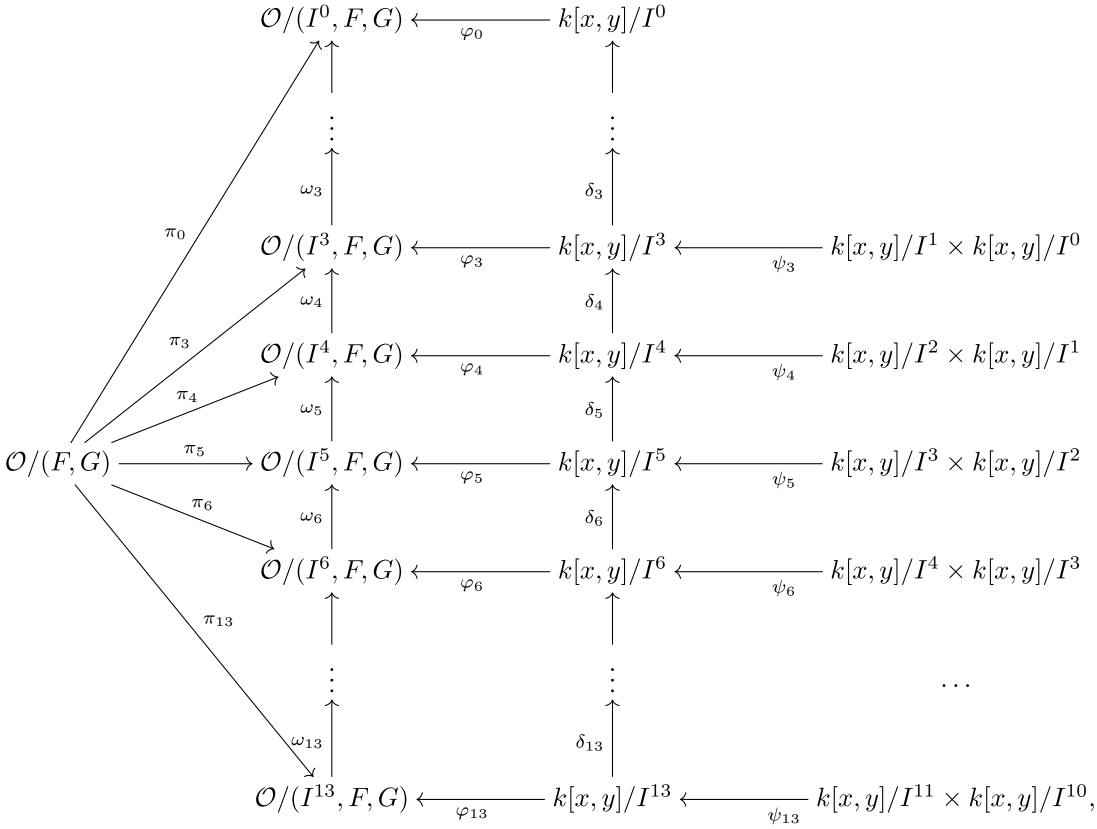
Let F and G be curves defined by the polynomials

$$\begin{aligned} F &= x^2 - y^7, \quad (12) \\ G &= x^3 - y^{10}. \quad (13) \end{aligned}$$

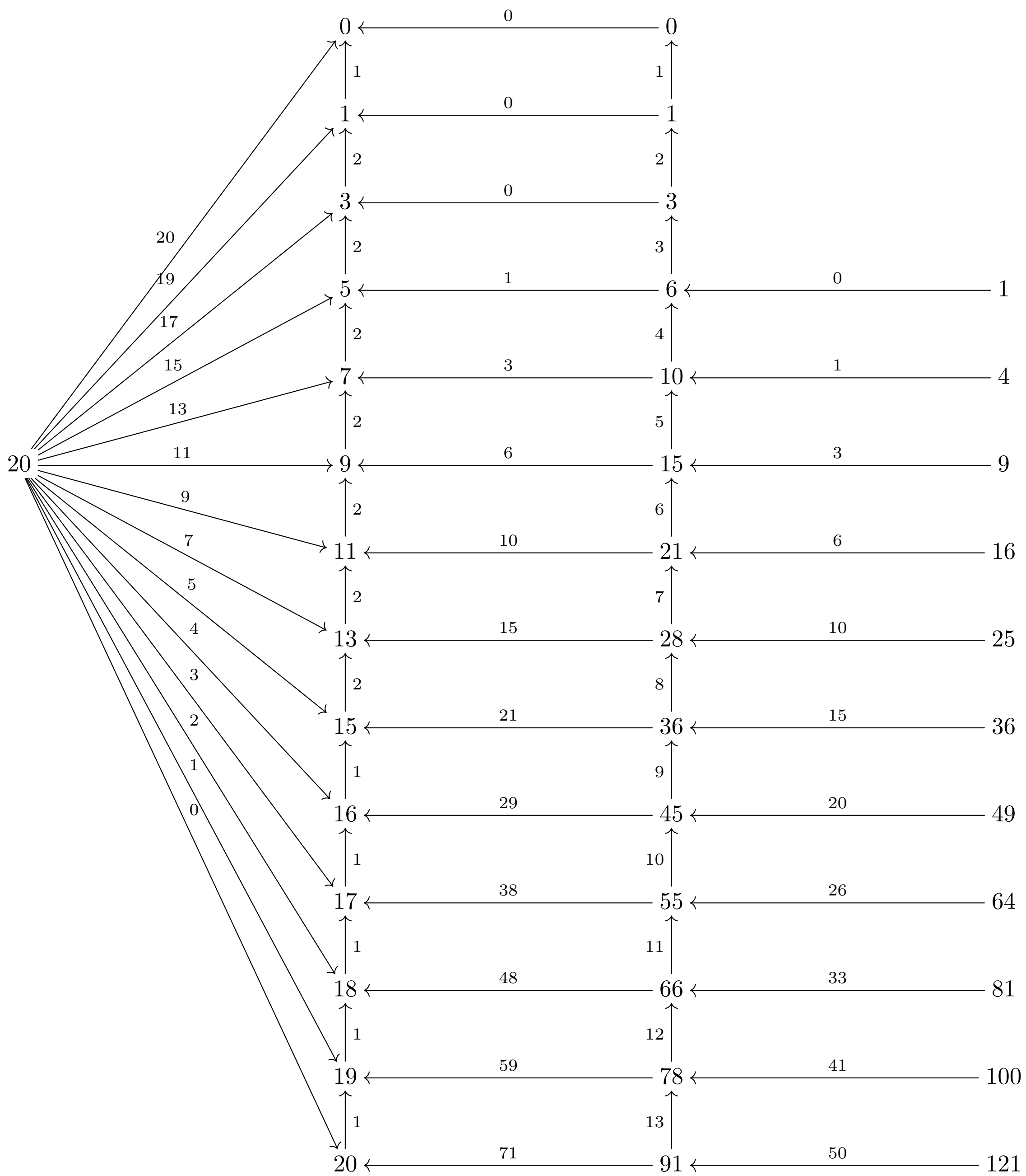


In this case, $m = 2$, $n = 3$ and we already know that $I_O(F,G) = 20$. The lowest possible value for z is

$z = 13$ (this has been calculated manually). The corresponding diagram looks like this:



Now the same diagram, but the vector spaces are replaced with their dimensions, and maps are replaced with dimensions of their kernels. (Note, that the maps ψ_i are not surjective.) I don't know if we ever need all of them, but I want to have them here anyway.



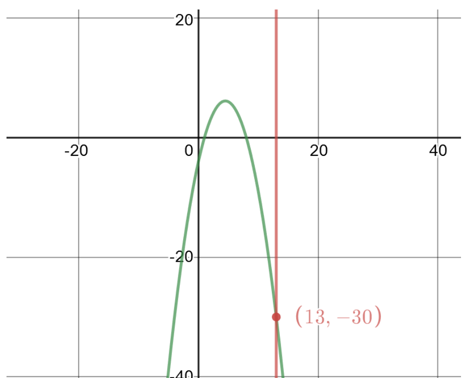
I have no conclusion for this, but I'm happy it works.
Anyway, we can check with our new pretty formula:

$$
\begin{aligned}
I_O(F,G) = \Xi_{m,n}(z) + \dim(\ker(\psi_z)) = z^2\left(-\frac{1}{2}\right) + z\left(m+n-\frac{1}{2}\right) + \frac{1}{2}(m+n-m^2-n^2) + \dim(\ker(\psi_z)) = \\
= \left(-\frac{1}{2}\right)13^2 + 13\left(2+3-\frac{1}{2}\right) + 2+3-4-9+50 = 20.
\end{aligned}
\tag{14}
$$

Nice.
And what does $\Xi_{m,n}(z)$ look like in this case?

$$
\Xi_{2,3}(z) = \left(-\frac{1}{2}\right)z^2 + \left(\frac{9}{2}\right)z - 4. \tag{15}
$$

The figure shows the intersection of $\Xi_{2,3}(z)$ with the line $z = 13$.



I wonder what Ξ looks like for other combinations of m and n.

3 A little about Ξ

The polynomial $\Xi_{m,n}(z) = z^2\left(-\frac{1}{2}\right) + z\left(m+n-\frac{1}{2}\right) + \frac{1}{2}(m+n-m^2-n^2)$ is a concave parabola for any combination of m and n. Its maximum is at the point $z = m+n-1/2$.
The actual graph is here: `https://www.desmos.com/calculator/9usvoa0nd3`

4 18.8 - UPDATE - a little about the sufficient values of z

Let F and G be curves defined by

$$\begin{aligned} F &= x^a - y^A, \quad (16) \\ G &= x^b - y^B. \quad (17) \end{aligned}$$

where $x = 0$ is their only common tangent at the origin. (therefore $a < A$ and $b < B$).

Then possible (not smallest possible) value for z is

$$z = \begin{cases} B + A - b - a + B + a - 1 & \text{if } I_O(F,G) = aB \\ B + A - b - a + b + A - 1 & \text{if } I_O(F,G) = bA \end{cases} \quad (18)$$

Current proof for this is ugly (= nonelegant). I hope I'll find a prettier one.

4.1 Some easier cases of z:

From now on, let z_0 be the lowest possible value of z.

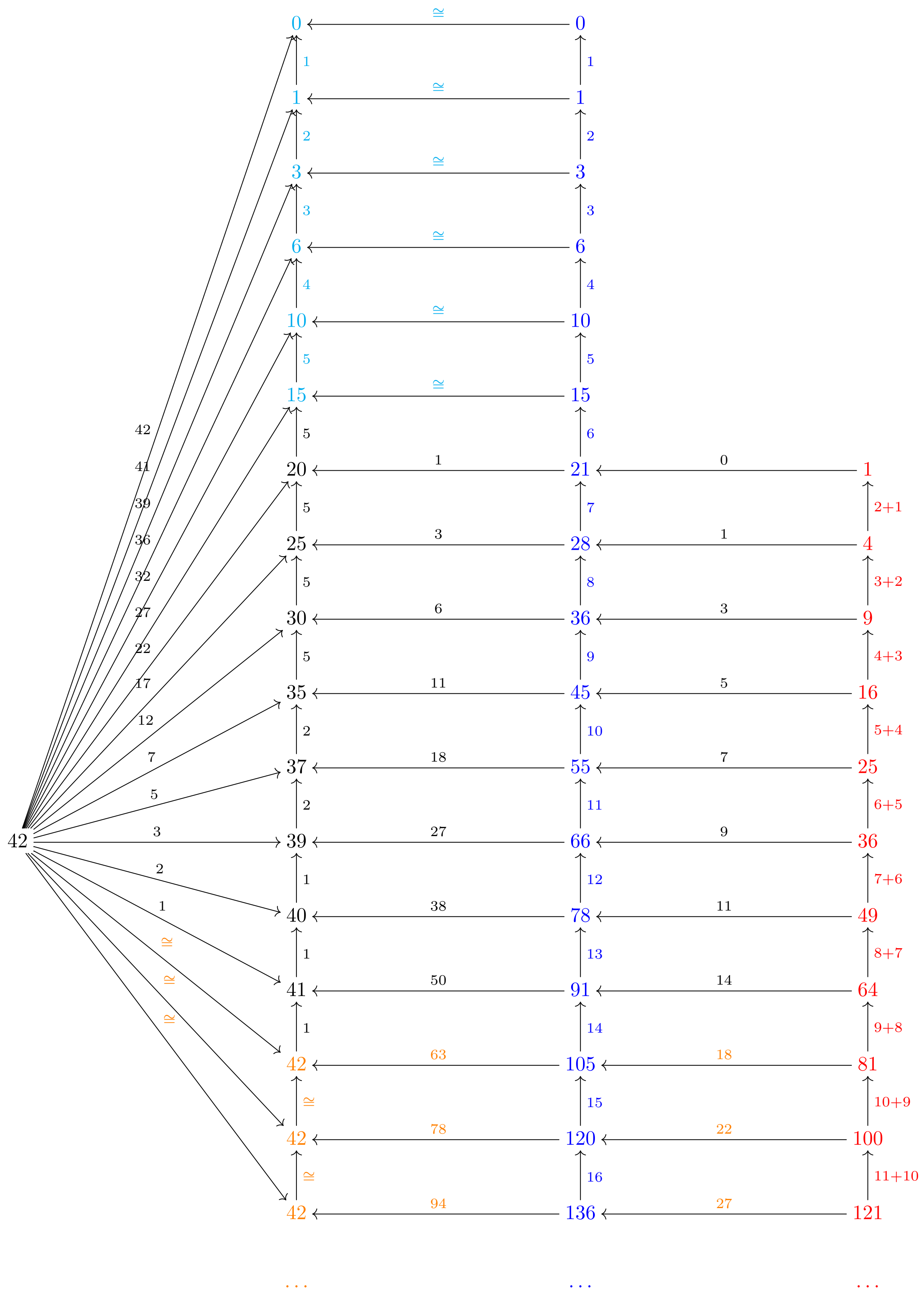
- If $a = b$ or $A = B$, then $(F, G, I^k) \sim (x^a, y^A, I^k)$, and $z_0 = a + A - 1$.
- If $b > a$ and $A > B$, then $(F, G, I^k) \sim (x^a, y^B, I^k)$, and $z_0 = a + B - 1$.
- I don't have the rest calculated yet

5 27.8. UPDATE - one more example

Let us do the same diagram for the curves

$$\begin{aligned} F &= x^5 - y^7, \quad (19) \\ G &= x^6 - y^{11}. \quad (20) \end{aligned}$$

This time, with color coding. The blue part does not depend on the choice of F and G (it depends only on the index of the row where it is positioned). The cyan part also depends only on the row index, but the number of cyan elements depend on m and n. The red part depends on row index, m and n. The orange part itself is most probably not very interesting, but it would be nice to determine where it starts (this is the number z_0). We can continue with the orange part for however long we want, but it will not bring us any more new information.



The blue, red and cyan spaces and maps have dimensions in obvious relation. In the orange part, we know that $(\dim(\ker(\psi_k)) - \dim(\ker(\psi_{k-1}))) = k - m - n$, and $(\dim(\ker(\varphi_k)) - \dim(\ker(\varphi_{k-1}))) = k$.

We expect the black parts to be the important parts.

QUESTION V:
Where does the cyan part end? Perhaps $\min(m,n)$?

6 UPDATE: 3.9.2020 - intermezzo

CH: Could $z_1 = \deg(F)\cdot\deg(G)$ be sufficient for every pair of curves F, G? Can we find such F and G, that $\mathcal{O}/(F,G,I^{z_1}) \cong \mathcal{O}/(F,G)$, but $\mathcal{O}/(F,G,I^{z_1-1}) \neq \mathcal{O}/(F,G)$?

CH: Can we find some connection between $\Xi_{m,n}$ and the Hilbert polynomial?

CH: Blow-ups of curves and their "trimmed" versions

CH: Dynkin diagram

CH: The derivative principle of $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

This is getting bleh again. I should make some pretty html version of this. i don't know :-/

CH - 20-09-02

7 UPDATE: 3.9.2020 - How does this structure behave in certain cases?

this update is basically a preparation, the next update (UPDATE 8.9.2020) is the result.

This is going to be a list of special cases of pairs of curves F and G and their behavior in the diagram. Hopefully i'll be able to cover as much cases as possible. I'll start with the very simple combinations of curves. Especially, we are interested in the sequence of degrees of the kernels of ω_i's. That means

$$\{0\} \cong \mathcal{O}/(I^0,F,G) \xleftarrow{\omega_1} \mathcal{O}/(I^1,F,G) \xleftarrow{\omega_2} \mathcal{O}/(I^2,F,G) \xleftarrow{\omega_3} \cdots \xleftarrow{\omega_{z_0}} \mathcal{O}/(I^{z_0},F,G) \cong \mathcal{O}/(F,G)$$

7.1 Properties of the sequence

Without loss of generality, let $F = F_m + F_{m+1} + \cdots$ and $G = G_n + G_{n+1} + \cdots$, where $m \le n$ and F_m and G_n are nonzero. Then

- The first m members of the sequence are equal to their index. For $i = 1,\dots,m$, $\dim(\ker(\omega_i)) = i$. This is because the here φ_i is always an isomorphism.
- Then, until the index $i = n$, $\dim(\ker(\omega_i)) = m$. This is because here, $\dim(\ker(\psi_i)) = 0$.
- After the index $i = m$, the sequence cannot grow anymore. So for $i > m$ we have $\dim(\ker(\omega_{i+1})) \le \dim(\ker(\omega_{i+1}))$.

  This needs a proper proof !
  UPDATE 24.9.: I think this ↑ can be proven from the existence of the maps ψ_i.

  Therefore once we reach $\omega_i = 0$, for all $j > i$, $\omega_j = 0$, and we are done.

So no matter what, we start with

$$
\begin{array}{ccc}
\mathcal{O}/(I^0, F, G) & \qquad & 0 \\
\uparrow \scriptstyle{\omega_1} & & \uparrow \scriptstyle{1} \\
\mathcal{O}/(I^1, F, G) & & 1 \\
\uparrow \scriptstyle{\omega_2} & & \uparrow \scriptstyle{2} \\
\mathcal{O}/(I^2, F, G) & & 3 \\
\uparrow \scriptstyle{\omega_3} & & \uparrow \scriptstyle{3} \\
\vdots & & \vdots \\
\uparrow \scriptstyle{\omega_m} & & \uparrow \scriptstyle{m} \\
\mathcal{O}/(I^m, F, G) & & \frac{1}{2}(m)(m+1) \\
\uparrow \scriptstyle{\omega_{m+1}} & & \uparrow \scriptstyle{m} \\
\mathcal{O}/(I^{m+1}, F, G) & & \frac{1}{2}(m)(m+1) + m \\
\uparrow \scriptstyle{\omega_{m+2}} & & \uparrow \scriptstyle{m} \\
\mathcal{O}/(I^{m+2}, F, G) & & \frac{1}{2}(m)(m+1) + 2m \\
\uparrow \scriptstyle{\omega_{m+3}} & & \uparrow \scriptstyle{m} \\
\vdots & & \vdots \\
\uparrow \scriptstyle{\omega_n} & & \uparrow \scriptstyle{m} \\
\mathcal{O}/(I^n, F, G) & & \frac{1}{2}(m)(m+1) + (n-m)m
\end{array}
$$

So the beginning of the sequence $\Omega = (\omega_1, \omega_2, \cdots)$ is always

$$1, \quad 2, \quad \cdots \quad m-1, \quad \underbrace{m, \quad m, \quad \cdots \quad m}_{n-m+1}$$

(the bottom curve indicates number of members). Sum of its members is $mn - \frac{1}{2}(m^2 - m)$, (which is less than mn). Now the geometry starts to be important.

QUESTION: Is there some kind of relationship between the sequences (F_1, F_2), (F_1, G), (F_2, G) and $(F_1 + F_2, G)$, $(F_1 \cdot F_2, G)$? Or something similar?

example E - 20-08-12 - $F = x^4 - y^{12}$, $G = x^7 - y^8$

example E - 20-08-10 - $F = x - y^8$, $G = x^5 - y^9$

7.2 Let F and G have no tangents in common

Then the sequence symmetrical with a mirror at $\frac{m+n}{2}$. Therefore it is

$$1, \quad \cdots \quad m-1, \quad \underbrace{m, \quad \cdots \quad m}_{n-m+1}, \quad m-1, \quad \cdots \quad 1, \quad 0, \quad 0, \quad \cdots$$

example E 20-08-27 - $F = x^3 - y^7$, $G = y^5 - x^8$

example E 20-08-15 - $F = (x-y)^2 + y^5$, $G = x^3 y - (x+y)^8$

7.3 Let F and G be curves which satisfy $I_O(F,G) = mn + t$

which is the minimal intersection multiplicity for two curves with t common tangents.
Then the sequence is almost the same as in the previous case, but on its way down, it contains the value t two times.

$$1, \quad \cdots \quad m-1, \quad \underbrace{m, \quad \cdots \quad m,}_{n-m+1} \quad m-1, \quad \cdots \quad t+1, \quad t, \quad t, \quad t-1, \quad \cdots \quad 1, \quad 0$$

I have a feeling it happened like this:

$$1, \quad \cdots \quad m-1, \quad \underbrace{m, \quad \cdots \quad m,}_{n-m+1} \quad m-1, \quad \cdots \quad t+1, \quad t, \quad t-1, \quad t-2, \quad \cdots \quad 1 \quad 0, \quad 0$$

$$\underbrace{+1 \quad +1 \quad \cdots \quad +1 \quad +1}_{t}$$

(the dotted arrows are not maps, they are just connections of the symbols on the paper)

example E 20-08-25 - $F = x^2 + F_3$, $G = xy^3 + G_5$

7.4 Let $m = 1$

Let F and G be curves defined by

$$F = x + F_2 + \cdots \tag{21}$$
$$G = xG'_{n-1} + G_{n+1} + \cdots . \tag{22}$$

Then our sequence Ω is

$$\underbrace{1, \quad \cdots \quad 1,}_{n+1} \quad \omega_{n+2}, \quad \omega_{n+3}, \quad \omega_{n+4}, \quad \cdots ,$$

where

- $\omega_{n+2} = 1$ if and only if $G_{n+1} - G'_{n-1}F_2$ is divisible by x. That means there is H_n (homogeneous of degree n or equal to zero), such that
$$G_{n+1} - G'_{n-1}F_2 = xH_n \tag{23}$$
Otherwise, it is 0 (along with all following members of the sequence).
- $\omega_{n+3} = 1$ if and only if there is H_{n+1} (homogeneous of degree $n+1$ or equal to zero), such that
$$G_{n+2} - G'_{n-1}F_3 - H_nF_2 = xH_{n+2}. \tag{24}$$
Otherwise, it is 0 (along with all following members of the sequence).
- $\omega_{n+4} = 1$ if and only if there is H_{n+2} (homogeneous of degree $n+2$ or equal to zero), such that
$$G_{n+3} - G'_{n-1}F_4 - H_nF_3 - H_{n+1}F_2 = xH_{n+2}. \tag{25}$$
Otherwise, it is 0 (along with all following members of the sequence).

example E 20-09-05 - general example

7.5 UPDATE 8.9.2020 - Conclusion of these cases

Okay, my hypothesis is that in general the sequence Ω looks like this: (where t is the number of common tangents at O)

$$1, \quad 2, \quad \cdots \quad m-1, \quad \underbrace{m, \quad \cdots \quad m,}_{n-m+1} \quad m-1, \quad \cdots \quad t, \quad t-1, \quad \cdots \quad 1, \quad 0, \quad 0, \quad \cdots$$

$$+? \quad \cdots \quad +? \quad +? \quad +? \quad \cdots$$

So basically the first row is fixed, but at the end, starting with the number $t-1$, the elements can be increased by some nonnegative integers. These increases depend on the intersection, but right now, I don't know how.

this needs a proper proof

By the way, sum of the first row is

$$\begin{aligned}\sum \Omega &= (1+2+\cdots+m-1) + m(n-m+1) + (m-1+m-2+\cdots+1) = \\ &= m(m-1) + m(n-(m-1)) = \\ &= mn,\end{aligned} \tag{26}$$

which is nice.

example E 20-09-08 - general example with only one tangent in common

example E 20-09-01 - example $F = x(x-y) + y^5$, $G = xy^3 + (x-y)^9$

8 20.1.2021: the $x^\alpha - y^\beta$ curves

Let F and G be curves defined by the polynomials

$$F = x^a - y^A, \tag{27}$$

$$G = x^b - y^B, \tag{28}$$

where $a < A$ and $b < B$ (that means each of these curves has the only tangent at O, the line $x = 0$, with the multiplicity a and b). Without loss of generality, let $a \leq b$.
In lot of cases, the sequence creates this neat (but a little boring) hill:

$$1, \quad 2, \quad \cdots \quad a-1, \quad \underbrace{a, \quad \cdots \quad a,}_{\frac{I}{a}-a+1} \quad a-1, \quad \cdots \quad 1, \quad 0, \quad 0, \quad \cdots$$

When it comes to numbers a, b, A, B, the general case split into several cases. Some of them are of the boring hill type.

add label

- Let $a = b < A < B$. Then

$$F = x^a - y^A, \tag{29}$$

$$G = x^a - y^{A+l}, \tag{30}$$

  and $I_O(F, G) = aA$. In this case $\mathcal{O}/(F, G, I^k) = \mathcal{O}/(x^a, y^A, I^k)$ for each k, therefore, the sequence is always of the boring hill type and $z_0 = a + A - 1$.

$$1, \quad 2, \quad \cdots \quad a-1, \quad \underbrace{a, \quad \cdots \quad a,}_{A-a+1} \quad a-1, \quad \cdots \quad 1, \quad 0, \quad 0, \quad \cdots$$

- Let $a < b < A = B$. Then

$$F = x^a - y^A, \tag{31}$$

$$G = x^{a+l} - y^A, \tag{32}$$

  and $I_O(F, G) = aA$. This case is similar to the previous one. Here $\mathcal{O}/(F, G, I^k) = \mathcal{O}/(x^a, y^A, I^k)$ for each k, therefore, the sequence is always of the boring hill type and $z_0 = a + A - 1$.

$$1, \quad 2, \quad \cdots \quad a-1, \quad \underbrace{a, \quad \cdots \quad a,}_{A-a+1} \quad a-1, \quad \cdots \quad 1, \quad 0, \quad 0, \quad \cdots$$

- Let $a < b < B < A$. Then

$$F = x^a - y^{B+L}, \tag{33}$$

$$G = x^{a+l} - y^B, \tag{34}$$

  and $I_O(F, G) = aB$. This case is similar to the previous one. Here $\mathcal{O}/(F, G, I^k) = \mathcal{O}/(x^a, y^B, I^k)$ for each k, therefore, the sequence is always of the boring hill type and $z_0 = a + B - 1$.

$$1, \quad 2, \quad \cdots \quad a-1, \quad \underbrace{a, \quad \cdots \quad a,}_{B-a+1} \quad a-1, \quad \cdots \quad 1, \quad 0, \quad 0, \quad \cdots$$

Now the leftover cases are

- $a < A < b < B$,
- $a < b < A < B$,
- $a < b = A < B$.

these are not as simple. They can result into the boring hill, but not necessarily.

When exactly does the boring hill happen?

9 Some examples

these need to be labeled

Each example starts with the equation $I_O(F,G) = mn + t + l$ in their corresponding numbers. First, some boring hills.

9.1 Let $F = x^2 - y^5$ and $G = x^2 - y^6$.

Then $10 = 2 \cdot 2 + 2 + 4$, $z_0 = 6$ and the Ω sequence is (boring hill):

$$1, \quad 2, \quad 2, \quad 2, \quad 2, \quad 1$$

9.2 Let $F = x^3 - y^7$ and $G = x^4 - y^7$.

Then $21 = 3 \cdot 4 + 3 + 6$, $z_0 = 9$ and the Ω sequence is (boring hill):

$$1, \quad 2, \quad 3, \quad 3, \quad 3, \quad 3, \quad 3, \quad 2, \quad 1$$

9.3 Let $F = x^5 - y^8$ and $G = x^{10} - y^{19}$.

Then $80 = 5 \cdot 10 + 5 + 25$, $z_0 = 20$ and the Ω sequence is (boring hill):

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \underbrace{\quad \cdots \quad}_{12} 5, \quad 4, \quad 3, \quad 2, \quad 1$$

Now some non-boring hills.

What is the shortest Ω sequence, which is not a boring hill?

9.4 Let $F = x^5 - y^8$ and $G = x^8 - y^{12}$.

Then $60 = 5 \cdot 8 + 5 + 15$, $z_0 = 17$ and the Ω sequence is:

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \underbrace{\quad \cdots \quad}_{7} 5, \quad 4, \quad 3, \quad 3, \quad 2, \quad 2, \quad 1$$

Now, since $I_O(F_1F_2, G) = I_O(F_1, G) + I_O(F_2, G)$, I've tried to do some examples to find out if we can see some kind of similar relationship in the Ω sequences, but with no luck so far.

9.5 Let $F = x^4 - y^6 = (x^2 - y^3)(x^2 + y^3) = F_1F_2$ and $G = x^2 - y^6$

Ok, I think it's time for some new notation, I need something like $\Omega_{F,G}$.

$$\Omega_{F_1,G} = \quad 1, \quad 2, \quad 2, \quad 1$$

$$\Omega_{F_2,G} = \quad 1, \quad 2, \quad 2, \quad 1$$

$$\Omega_{F_1F_2,G} = \quad 1, \quad 2, \quad 2, \quad 2, \quad 2, \quad 2, \quad 1$$

9.6 Let $F = x^6 - y^{10} = (x^3 - y^5)(x^3 + y^5) = F_1F_2$ and $G = x^7 - y^{12}$

Then

$$\Omega_{F_1,G} = \quad 1, \quad 2, \quad 3, \underbrace{\quad\cdots\quad}_{9} 3, \quad 2, \quad 1, \quad 1, \quad 1$$

$$\Omega_{F_2,G} = \quad 1, \quad 2, \quad 3, \underbrace{\quad\cdots\quad}_{9} 3, \quad 2, \quad 1, \quad 1, \quad 1$$

$$\Omega_{F_1F_2,G} = \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \underbrace{\quad\cdots\quad}_{6} 6, \quad 5, \quad 3, \quad 2, \quad 1, \quad 1, \quad 1, \quad 1, \quad 1$$

Let $F = x^2 - y^3$, $G = x^3 - y^5$

Then $9 = 2 \cdot 3 + 2 + 1$, $z_0 = 6$ and the Ω sequence is:

$$1, \quad 2, \quad 2, \quad 2, \quad 1, \quad 1$$

Also I think this is the shortest non-boring hill one of the cases $F = x^a - y^A$, $G = x^b - y^B$.

10 23.2.2021: A slightly different point of view

I'm beginning to see the whole problem to be not of a problem of kernels, but a problem of differences between the kernels.
First of all, let us revisit the sequence Ω. Let $F = F_m + \cdots$ and $G = G_n + \cdots$, with $m \leq n$. The "ground" of the sequence can be understood as this:

$$1, \overbrace{2, \quad \cdots \quad m-1, \quad m, \underbrace{\quad\cdots\quad}_{n-m+1} m,}^{\text{this part is fixed no matter what}} \quad m-1, \quad m-2, \quad \cdots \quad 3, \quad 2, \quad 1, \quad 0, \quad -1 \quad -2 \quad -3 \quad \cdots$$

$$\begin{array}{ccccccccccc} \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ +? & +? & \cdots & +? & +? & +? & +? & +? & +? & +? & \cdots \end{array}$$

The first (fixed) part looks like above for any pair of curves F and G and therefore is not very interesting. The rest of the sequence can be increased by some integers, depending on the curves. This (nonfixed) part comes from the polynomial $\Xi_{m,n}(z)$ defined as

$$\Xi_{m,n}(z) = z^2\left(-\frac{1}{2}\right) + z\left(m + n - \frac{1}{2}\right) + \frac{1}{2}(m + n - m^2 - n^2). \tag{35}$$

Let us remind that

$$\begin{aligned} \mathcal{O}/(F, G, I^z) &= \Xi_{m,n}(z) + \dim(\ker(\psi_z)) \quad &\text{for } z \geq n \\ I_O(F, G) = \mathcal{O}/(F, G, I^z) &= \Xi_{m,n}(z) + \dim(\ker(\psi_z)) \quad &\text{for } z \geq z_0. \end{aligned} \tag{36}$$

For our purposes, the polynomial $\Xi_{m,n}(z)$ makes sense only for $z \geq n$, because it is defined as $\Xi_{m,n}(z) = \sum_{i=1}^{z} \dim(\ker(\delta_i)) - (\dim(k[x,y]/I^{z-m} \times k[x,y]/I^{z-n}))$

The integers from the "ground" are actually the differences $(\Xi_{m,n}(z) - \Xi_{m,n}(z-1))$. Now we can say what the integers are increased by. Let $\Psi_i = \dim(\ker(\psi_i))$. Then

$$1, \overbrace{\quad \cdots \quad m-1, \quad m, \underbrace{\quad\cdots\quad}_{n-m+1} m,}^{\text{fixed part}} \quad m-1, \quad m-2, \quad m-3 \quad \cdots$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \\ +(\Psi_{n+1} - 0) & +(\Psi_{n+2} - \Psi_{n+1}) & +(\Psi_{n+3} - \Psi_{n+2}) & \cdots \end{array}$$

Maybe I should add some "difference" notation, for the sake of shorter text. For now, let $\Delta_i = \Psi_i - \Psi_{i-1}$. Since $\Psi_n = \dim(\ker(\psi_n)) = 0$, we can write this as

$$\begin{array}{ccccccccc} \overbrace{1, \cdots m, \underbrace{\cdots m,}_{n-m+1}}^{\text{fixed part}} & m-1, & m-2, & \cdots & 1, & 0, & -1 & -2 & \cdots \\ & \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow & \uparrow & \\ & +\Delta_{n+1} & +\Delta_{n+2} & \cdots & +\Delta_{n+m-1} & +\Delta_{n+m} & +\Delta_{n+m+1} & +\Delta_{n+m+2} & \cdots \end{array}$$

So we are kinda back at the question of the dimension kernels of $\psi_z : k[x,y]/I^{z-m} \times k[x,y]/I^{z-n} \longrightarrow k[x,y]/I^z$ (where $\psi_z(A,B) = AF - BG$).
This time it might be a little bit different, because we are not that interested in the kernel itself, but in the difference of the dimensions of kernels ψ_{z+1} and ψ_z. This is very important computation-wise.

Now what do we know about these kernels and their differences so far?

First of all, if F and G have no tangents in common, all the kernels (and also their differences) are equal to zero, up till the index $n+m+1$. Then the kernels are:

$$\begin{aligned} \ker(\psi_{m+n-1}) =& \{0\} \\ \ker(\psi_{m+n}) =& \{0\} \\ \ker(\psi_{m+n+1}) =& (A_0G_n, A_0F_m) \\ \ker(\psi_{m+n+2}) =& (A_0(G_n + G_{n+1}) + A_1G_n, A_0(F_m + F_{m+1}) + A_1F_m) \\ \ker(\psi_{m+n+3}) =& (A_0(G_n + G_{n+1} + G_{n+2}) + A_1(G_n + G_{n+1}) + A_2G_n, \\ & A_0(F_m + F_{m+1} + F_{m+2}) + A_1(F_m + F_{m+1}) + A_2F_m) \\ & \cdots, \end{aligned} \tag{37}$$

where A_i si either a homogeneous polynomials of degree i of equal to zero. Dimensions of these subspaces are

$$\begin{aligned} \dim(\ker(\psi_{m+n-1})) = \Psi_{m+n-1} =& 0 \\ \dim(\ker(\psi_{m+n})) = \Psi_{m+n} =& 0 \\ \dim(\ker(\psi_{m+n+1})) = \Psi_{m+n+1} =& 1 \\ \dim(\ker(\psi_{m+n+2})) = \Psi_{m+n+2} =& 1 + 2 = 3 \\ \dim(\ker(\psi_{m+n+3})) = \Psi_{m+n+3} =& 1 + 2 + 3 = 6 \\ & \cdots, \end{aligned} \tag{38}$$

and their differences are

$$\begin{aligned} \Delta_{n+m} = \Psi_{m+n} - \Psi_{m+n-1} =& 0 - 0 = 0 \\ \Delta_{n+m+1} = \Psi_{m+n+1} - \Psi_{m+n} =& 1 - 0 = 1 \\ \Delta_{n+m+2} = \Psi_{m+n+2} - \Psi_{m+n+1} =& 3 - 1 = 2 \\ \Delta_{n+m+3} = \Psi_{m+n+3} - \Psi_{m+n+2} =& 6 - 3 = 3 \\ \Delta_{n+m+4} = \Psi_{m+n+4} - \Psi_{m+n+3} =& 10 - 6 = 4 \\ & \cdots, \end{aligned} \tag{39}$$

which is just what we need to even up the negative integers from the sequence Ω. So for this case we get

$$\begin{array}{ccccccccccc} \overbrace{1, \cdots m, \underbrace{\cdots m,}_{n-m+1}}^{\text{fixed part}} & m-1, & m-2, & \cdots & 1, & 0, & -1 & -2 & -3 & -4 & \cdots \\ & \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ & +0 & +0 & \cdots & +0 & +0 & +1 & +2 & +3 & +4 & \cdots \end{array}$$

which is just the most simple basic boring hill.
If F and G do have some tangents in common, the solutions above will apply too, they are just not ALL of the solutions. They will be included in the corresponding kernels as subspaces. So the increase illustrated at the sequence above is basically the minimal increase.
If K_i is some subspace of $\ker(\psi_i)$, then every $\ker(\psi_j)$ $(j > i)$ will contain a corresponding subspace K_j, where

$K_{i+1} = D_1 K_i$ (D_1 is either homogeneous polynomial of degree 1 or equal to 0). Obviously $\dim(K_{i+1}) = \dim(K_i) + 1$. Maybe a simpler way of putting is by showing that if $K \in \ker(\psi_i)$, then $D_1 K \in \ker(\psi_{i+1})$, $D_2 K \in \ker(\psi_{i+2})$, $D_3 K \in \ker(\psi_{i+3})$, etc (where D_i is homogeneous polynomial of degree i). Difference of their dimensions $(\dim(D_{i+1}K) - \dim(D_i K))$ is always equal to 1. Therefore every new subspace which occurs at some point of the algorithm will start a full row of ones that starts at this points that continues to infinity. This is illustrated in the following example.

11 14.5.2021: Demonstration of the idea above on an example

Example 1. *Let*

$$F = x^2 - y^5 \tag{40}$$
$$G = x^4 - y^7 \tag{41}$$

Then the kernels of the maps ψ_i ($\psi_i : k[x,y]/I^{i-n} \times k[x,y]/I^{i-m} \longrightarrow k[x,y]/I^i$) *are*

$$\begin{aligned}
\ker(\psi_3) &= (0,0)\\
\ker(\psi_4) &= (0,0)\\
\ker(\psi_5) &= D_0(x^2,1)\\
\ker(\psi_6) &= D_0(x^2,1) + D_1(x^2,1)\\
\ker(\psi_7) &= D_0(x^2,1) + D_1(x^2,1) + D_2(x^2,1)\\
\ker(\psi_8) &= D_1(x^2,1) + D_2(x^2,1) + D_3(x^2,1)\\
\ker(\psi_9) &= D_2(x^2,1) + D_3(x^2,1) + D_4(x^2,1)\\
\ker(\psi_{10}) &= D_3(x^2,1) + D_4(x^2,1) + D_5(x^2,1) + E_0(x^4-y^7, x^2-y^5)\\
\ker(\psi_{11}) &= D_4(x^2,1) + D_5(x^2,1) + D_6(x^2,1) + E_0(x^4-y^7, x^2-y^5) + E_1(x^4-y^7, x^2-y^5)\\
\ker(\psi_{12}) &= D_4(x^2,1) + D_5(x^2,1) + D_6(x^2,1) + E_0(x^4-y^7, x^2-y^5) + E_1(x^4-y^7, x^2-y^5) + E_2(x^4-y^7, x^2-y^5)\\
&\dots
\end{aligned} \tag{42}$$

and the Omega sequence looks like this:

1	2	2	2	2	2	2	1	0	0	0	0	0	0	…
=	=	=	=	=	=	=	=	=	=	=	=	=	=	…
1	2	2	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8	…
				1	1	1	1	1	1	1	1	1	1	…
					1	1	1	1	1	1	1	1	1	…
						1	1	1	1	1	1	1	1	…
									1	1	1	1	1	…
										1	1	1	1	…
											1	1	1	…
													…	

where the first row is the final sequence, second row is the ground(non-kernel contributions) and the other rows are kernel contributions. Each element of the first row is the sum of the elements in its column, below it.
We can split the kernel contributions of this sequence into the boring ones (these exist in every intersection and

depend only on m and n) and the interesting ones (these depend on the properties of the intersection)

1	2	2	2	2	2	2	1	0	0	0	0	0	0	$\cdots$
=	=	=	=	=	=	=	=	=	=	=	=	=	=	$\cdots$
1	2	2	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8	$\cdots$
				1	1	1	1	1	1	1	1	1	1	$\cdots$
					1	1	1	1	1	1	1	1	1	$\cdots$
						1	1	1	1	1	1	1	1	$\cdots$
									1	1	1	1	1	$\cdots$
										1	1	1	1	$\cdots$
											1	1	1	$\cdots$
													$\cdots$	

Of course, the sum of the first row is equal to the intersection multiplicity of these two curves. $I_O(F,G) = 1+2+2+2+2+2+2+1 = 14$.

more examples do exist on paper

In the project of Disseration we defined $\tau_k = \max\{\deg(\gcd(F_m, \dots, F_{m+k}, G_n, \dots, G_{n+k}))\}$ (which have the property $I_O(F,G) \geq mn + \tau_0 + \cdots + \tau_m$). We already in what way the τ_k appear here.

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also i think this could be made into an algorithm