

3.1.2022: Maybe making these into exact sequences (starting with 0) can be a good idea (an example)

Let F and G be defined by

$$\begin{aligned} F &= x^3 - y^5, \\ G &= x^5 - y^8. \end{aligned} \tag{1}$$

link na graf: <https://www.desmos.com/calculator/lvywu1yc6n>

Therefore $m = 3$ and $n = 5$. Then $\ker \psi_0 = \dots = \ker \psi_5 = (0, 0)$ and

$$\begin{aligned} \ker \psi_6 &= D_0(x^2, 1) \\ \ker \psi_7 &= D_1(x^2, 1) + D_0(x^2, 1) \\ \ker \psi_8 &= D_2(x^2, 1) + D_1(x^2, 1) \\ \ker \psi_9 &= D_3(x^2, 1) + D_2(x^2, 1) + D_0(x^3 + y^5, x) \\ \ker \psi_{10} &= D_4(x^2, 1) + D_3(x^2, 1) + D_1(x^3 + y^5, x) \\ \ker \psi_{11} &= D_5(x^2, 1) + D_4(x^2, 1) + D_2(x^3 + y^5, x) + D_0(x^4 + x^2y^3 + xy^5, x^2 + y^3) \\ \ker \psi_{12} &= D_6(x^2, 1) + D_5(x^2, 1) + D_3(x^3 + y^5, x) + D_1(x^4 + x^2y^3 + xy^5, x^2 + y^3) \\ \ker \psi_{13} &= D_7(x^2, 1) + D_6(x^2, 1) + D_4(x^3 + y^5, x) + D_2(x^4 + x^2y^3 + xy^5, x^2 + y^3) + D_0(x^5 - y^8, x^3 - y^5) \\ \ker \psi_{14} &= D_8(x^2, 1) + D_7(x^2, 1) + D_5(x^3 + y^5, x) + D_3(x^4 + x^2y^3 + xy^5, x^2 + y^3) + D_1(x^5 - y^8, x^3 - y^5) + D_0(x^5 - y^8, x^3 - y^5) \\ \ker \psi_{15} &= \dots \end{aligned} \tag{2}$$

The ψ table is

$$\begin{array}{cccccccccccccccc} 1 & 2 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & \dots \\ = & = & = & = & = & = & = & = & = & = & = & = & = & = & \dots \\ 1 & 2 & 3 & 3 & 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 & -5 & -6 & \dots \\ & & & & & & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ & & & & & & & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \dots \\ & & & & & & & & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \dots \\ & & & & & & & & & & 1 & \cdot & \cdot & \cdot & \dots \\ & & & & & & & & & & & \cdot & \cdot & \cdot & \dots \\ & & & & & & & & & & & & \cdot & \cdot & \dots \\ & & & & & & & & & & & & & \cdot & \dots \end{array} \tag{3}$$

Therefore

$$I_O(F, G) = mn + \sum a_i = 3 \cdot 5 + 3 + 3 + 2 + 1 = 24. \tag{4}$$

So we can construct this diagram where each row is an exact sequence. Here, I is the ideal (x, y) , and $\mathcal{S}_i = I^i/I^{i+1}$. The maps φ_i (and also all the vertical maps) are canonical, and $\psi_i(A, B) = FA - BG$. The maps γ_i are defined below.

$$\begin{array}{ccccccc}
0 & \xrightarrow{\psi_0} & k[x, y]/I^0 & \xrightarrow{\varphi_0} & k[x, y]/(I^0, F, G) & \longrightarrow & 0 \\
& & \uparrow 1 & & \uparrow 1 & & \\
0 & \xrightarrow{\psi_1} & k[x, y]/I^1 & \xrightarrow{\varphi_1} & k[x, y]/(I^1, F, G) & \longrightarrow & 0 \\
& & \uparrow 2 & & \uparrow 2 & & \\
0 & \xrightarrow{\psi_2} & k[x, y]/I^2 & \xrightarrow{\varphi_2} & k[x, y]/(I^2, F, G) & \longrightarrow & 0 \\
& & \uparrow 3 & & \uparrow 3 & & \\
0 & \xrightarrow{\psi_3} & k[x, y]/I^3 & \xrightarrow{\varphi_3} & k[x, y]/(I^3, F, G) & \longrightarrow & 0 \\
& & \uparrow 4 & & \uparrow 3 & & \\
0 & \xrightarrow{\gamma_4} & k[x, y]/I^1 & \xrightarrow{\psi_4^0} & k[x, y]/I^4 & \xrightarrow{\varphi_4^{1-0=1}} & k[x, y]/(I^4, F, G) \longrightarrow 0 \\
& & \uparrow 1+0=1 & & \uparrow 10 & & \uparrow 9 \\
& & \uparrow 2+0 & & \uparrow 5 & & \uparrow 3 \\
0 & \xrightarrow{\gamma_5} & k[x, y]/I^2 \times k[x, y]/I^0 & \xrightarrow{\psi_5^0} & k[x, y]/I^5 & \xrightarrow{\varphi_5^{3-0=3}} & k[x, y]/(I^5, F, G) \longrightarrow 0 \\
& & \uparrow 3+0=3 & & \uparrow 15 & & \uparrow 12 \\
& & \uparrow 3+1 & & \uparrow 6 & & \uparrow 3 \\
0 & \xrightarrow{\gamma_6} & k[x, y]/I^3 \times k[x, y]/I^1 & \xrightarrow{\psi_6^1} & k[x, y]/I^6 & \xrightarrow{\varphi_6^{7-1=6}} & k[x, y]/(I^6, F, G) \longrightarrow 0 \\
& & \uparrow 6+1=7 & & \uparrow 21 & & \uparrow 15 \\
& & \uparrow 4+2 & & \uparrow 7 & & \uparrow 3 \\
0 & \xrightarrow{\gamma_7} & k[x, y]/I^4 \times k[x, y]/I^2 & \xrightarrow{\psi_7^3} & k[x, y]/I^7 & \xrightarrow{\varphi_7^{13-3=10}} & k[x, y]/(I^7, F, G) \longrightarrow 0 \\
& & \uparrow 10+3=13 & & \uparrow 28 & & \uparrow 18 \\
& & \uparrow 5+3 & & \uparrow 8 & & \uparrow 2 \\
0 & \xrightarrow{\gamma_8} & k[x, y]/I^5 \times k[x, y]/I^3 & \xrightarrow{\psi_8^5} & k[x, y]/I^8 & \xrightarrow{\varphi_8^{21-5=16}} & k[x, y]/(I^8, F, G) \longrightarrow 0 \\
& & \uparrow 15+6=21 & & \uparrow 36 & & \uparrow 20 \\
& & \uparrow 6+4 & & \uparrow 9 & & \uparrow 2 \\
0 & \xrightarrow{\gamma_9} & k[x, y]/I^6 \times k[x, y]/I^4 & \xrightarrow{\psi_9^8} & k[x, y]/I^9 & \xrightarrow{\varphi_9^{31-8=23}} & k[x, y]/(I^9, F, G) \longrightarrow 0 \\
& & \uparrow 21+10=31 & & \uparrow 45 & & \uparrow 22 \\
& & \uparrow 7+5 & & \uparrow 10 & & \uparrow 1 \\
0 & \xrightarrow{\gamma_{10}} & k[x, y]/I^7 \times k[x, y]/I^5 & \xrightarrow{\psi_{10}^{11}} & k[x, y]/I^{10} & \xrightarrow{\varphi_{10}^{43-11=32}} & k[x, y]/(I^{10}, F, G) \longrightarrow 0 \\
& & \uparrow 28+15=43 & & \uparrow 55 & & \uparrow 23 \\
& & \uparrow 8+6 & & \uparrow 11 & & \uparrow 1 \\
0 & \xrightarrow{\gamma_{11}} & k[x, y]/I^8 \times k[x, y]/I^6 & \xrightarrow{\psi_{11}^{15}} & k[x, y]/I^{11} & \xrightarrow{\varphi_{11}^{57-15=42}} & k[x, y]/(I^{11}, F, G) \longrightarrow 0 \\
& & \uparrow 36+21=57 & & \uparrow 66 & & \uparrow 24 \\
& & \uparrow 9+7 & & \uparrow 12 & & \uparrow 0 \\
0 & \xrightarrow{\gamma_{12}} & k[x, y]/I^9 \times k[x, y]/I^7 & \xrightarrow{\psi_{12}^{19}} & k[x, y]/I^{12} & \xrightarrow{\varphi_{12}^{73-19=54}} & k[x, y]/(I^{12}, F, G) \longrightarrow 0 \\
& & \uparrow 45+28=73 & & \uparrow 78 & & \uparrow 24 \\
& & \uparrow 10+8 & & \uparrow 13 & & \uparrow 0 \\
0 & \xrightarrow{\gamma_{13}} & k[x, y]/I^{10} \times k[x, y]/I^8 & \xrightarrow{\psi_{13}^{24}} & k[x, y]/I^{13} & \xrightarrow{\varphi_{13}^{91-24=67}} & k[x, y]/(I^{13}, F, G) \longrightarrow 0 \\
& & \uparrow 55+36=91 & & \uparrow 91 & & \uparrow 24 \\
& & \uparrow 11+9 & & \uparrow 14 & & \uparrow 0 \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

The γ maps are obviously based on kernels of ψ maps.

A sequence
depends only
on m and n
units + dots

$$\begin{aligned}
\gamma_6: & \quad D_0 \longrightarrow D_0(x^2, 1) & = (D_0x^2, D_0) \\
\gamma_7: & \quad (D_0, D_1) \longrightarrow D_1(x^2, 1) + D_0(x^2, 1) & = (D_1x^2 + D_0x^2, D_1 + D_0) \\
\gamma_8: & \quad (D_1, D_2) \longrightarrow D_2(x^2, 1) + D_1(x^2, 1) & = \dots \\
\gamma_9: & \quad (D_0, D_2, D_3) \longrightarrow D_3(x^2, 1) + D_2(x^2, 1) + D_0(x^3 + y^5, x) \\
\gamma_{10}: & \quad (D_1, D_3, D_4) \longrightarrow D_4(x^2, 1) + D_3(x^2, 1) + D_1(x^3 + y^5, x) \\
\gamma_{11}: & \quad (D_0, D_2, D_4, D_5) \longrightarrow D_5(x^2, 1) + D_4(x^2, 1) + D_2(x^3 + y^5, x) + D_0(x^4 + x^2y^3 + xy^5, x^2 + y^3) \\
\gamma_{12}: & \quad (D_1, D_3, D_5, D_6) \longrightarrow D_6(x^2, 1) + D_5(x^2, 1) + D_3(x^3 + y^5, x) + D_1(x^4 + x^2y^3 + xy^5, x^2 + y^3) \\
\gamma_{13}: & \quad (D_0, D_2, D_4, D_6, D_7) \longrightarrow D_7(x^2, 1) + D_6(x^2, 1) + D_4(x^3 + y^5, x) + D_2(x^4 + x^2y^3 + xy^5, x^2 + y^3) + \\
& \quad \quad \quad + D_0(x^5 - y^8, x^3 - y^5) \\
\gamma_{14}: & \quad (D_0, D_1, D_3, D_5, D_7, D_8) \longrightarrow D_8(x^2, 1) + D_7(x^2, 1) + D_5(x^3 + y^5, x) + D_3(x^4 + x^2y^3 + xy^5, x^2 + y^3) + \\
& \quad \quad \quad + D_1(x^5 - y^8, x^3 - y^5) + D_0(x^5 - y^8, x^3 - y^5) \\
\gamma_{15}: & \quad \dots
\end{aligned} \tag{5}$$