30.6.2023 examples

Let $T = ax - by$ be a tangent. We already have conditions for which $T \mid {}_0L$ and $T \mid {}_1L$. Now we would like to find out when does $T \mid {}_2L$ and connect it to the geometry of the situation (probably the contribution of the tangent T to the intersection multiplicity)

example 1

Let $F_m = xy$ and $\mathcal{L} = ({}_0L, {}_1L, {}_2L, {}_3L, \cdots) = (xy, xy, xy, x, \cdots)$. Then the polynomials F and G are of the form

$$F = xy + F_3 + F_4 + F_5 + \cdots$$
$$G = xyZ_{n-2} + [xyZ_{n-1} + F_3Z_{n-2}] + [xyZ_n + F_3Z_{n-1} + F_4Z_{n-2}] + [xZ_{n+2} + F_3Z_n + F_4Z_{n-1} + F_5Z_{n-2}] + \cdots$$

For any F_i, Z_i, such that $y \nmid Z_{n+2}$.

example 2

Let $F_m = xy$ and $\mathcal{L} = ({}_0L, {}_1L, {}_2L, \cdots) = (xy, xy, {}_2L, \cdots)$. $(y \nmid {}_2L)$ Then the polynomials F and G are of the form

$$F = xy + F_3 + F_4 + \cdots$$
$$G = xyZ_{n-2} + [xyZ_{n-1} + F_3Z_{n-2}] + [Z_{n+2} + F_3Z_{n-1} + F_4Z_{n-2}] + \cdots$$

For any F_i, Z_i, such that $y \nmid Z_{n+2}$.
Alternatively, we can make this into a set of conditions.

- $\gcd(F_2, G_n) = xy \quad \Longrightarrow \quad {}_0L = xy$
- $xy \mid \left(G_{n+1} - \frac{1}{xy}G_nF_3\right) \quad \Longrightarrow \quad {}_1L = xy$
- $y \nmid \left(G_{n+2} - \frac{1}{xy}F_3\left(G_{n+1} - \frac{1}{xy}G_nF_3\right) - \frac{1}{xy}G_nF_4\right) \quad \Longrightarrow \quad y \nmid {}_2L$

example 3

Let $F_m = {}_0L$, which means that $F = {}_0L + F_{m+1} + F_{m+2} + \cdots$. Let G be a general polynomial $G = G_n + G_{n+1} + \cdots$. Then

- ${}_0L = \gcd(F_m, G_n) = F_m$
- ${}_1\Upsilon = \Upsilon_{m+n-a_0+1} = G_{n+1} - \frac{G_n}{F_m}F_{m+1} = \frac{1}{F_m}(F_mG_{n+1} - G_nF_{m+1})$. Therefore

$$ {}_1L = \gcd({}_0L, {}_1\Upsilon) = \gcd\left(F_m, G_{n+1} - \frac{G_n}{F_m}F_{m+1}\right) = L_{a_1}$$

- ${}_2\Upsilon = \Upsilon_{m+n-a_1+2} = \frac{1}{F_m}\frac{1}{L_{a_1}}(F_mG_{n+1} - G_nF_{m+1})F_{m+1} + \frac{L_{a_0}}{L_{a_1}}\frac{F_m}{L_{a_0}}G_{n+2} - \frac{L_{a_0}}{L_{a_1}}\frac{G_n}{L_{a_0}}F_{m+2} =$
  . $= \frac{1}{L_{a_1}}\left(\left(G_{n+1} - \frac{G_n}{F_m}F_{m+1}\right)F_{m+1} + F_mG_{n+2} - G_nF_{m+2}\right) =$
  . $= \frac{1}{L_{a_1}}\left(G_{n+1}F_{m+1} - \frac{G_n}{F_m}F_{m+1}^2 + F_mG_{n+2} - G_nF_{m+2}\right),$

  and
  $$ {}_2L = \gcd({}_1L, {}_2\Upsilon) = \gcd\left(L_{a_1}, \frac{1}{L_{a_1}}\left(\left(G_{n+1} - \frac{G_n}{F_m}F_{m+1}\right)F_{m+1} + F_mG_{n+2} - G_nF_{m+2}\right)\right) = L_{a_2}$$

In case of $L_{a_1} \mid F_{m+1}$, we can simplify this to

- ${}_2\Upsilon = \Upsilon_{m+n-a_1+2} = \frac{1}{L_{a_1}}(F_mG_{n+2} - G_nF_{m+2})$, therefore

$$ {}_2L = \gcd({}_1L, {}_2\Upsilon) = \gcd\left(L_{a_1}, \frac{1}{L_{a_1}}(F_mG_{n+2} - G_nF_{m+2})\right) = L_{a_2}$$

The polynomials F and G are in the form

$$F = F_m + F_{m+1} + F_{m+2} + \cdots$$
$$G = (G_n) + (G_{n+1}) + (G_{n+2}) + \cdots =$$
$$= (F_mZ_{n-m}) + (Z_{n-m}F_{m+1} + L_{a_1}H_{n-a_1+1}) +$$
$$+ \left(\frac{L_{a_1}}{F_m}\left(H_{n-a_1+1}\left(\beta_{m-a_1+1}L_{a_1} - F_{m+1}\right) - H_{m+n-a_1-a_2+2}L_{a_2}\right) - Z_{n-m}F_{m+2}\right)$$

such that

- $L_{a_1} \mid F_m \wedge L_{a_2} \mid L_{a_1}$
- $\gcd\left(H_{n+1-a_1}, \frac{F_m}{L_{a_1}}\right) = 1 \wedge \gcd\left(H_{m+n-a_1-a_2+2}, \frac{L_{a_1}}{L_{a_2}}\right) = 1$
- $\frac{F_m}{L_{a_1}} \mid (H_{n-a_1+1}(\beta_{m-a_1+1}L_{a_1} - F_{m+1}) - H_{m+n-a_1-a_2+2}L_{a_2})$

example 4

$\mathcal{L}$ polynomials in general case: Let $F = F_m + F_{m+1} + \cdots$ and $G = G_n + G_{n+1} + \cdots$. Then

- ${}_0L = \gcd(F_m, G_n) = L_{a_0}$
- ${}_1\Upsilon = \Upsilon_{m+n-a_0+1} = \frac{F_m}{L_{a_0}}G_{n+1} - \frac{G_n}{L_{a_0}}F_{m+1} = \frac{1}{L_{a_0}}(F_mG_{n+1} - G_nF_{m+1})$. Therefore

$$ {}_1L = \gcd({}_0L, {}_1\Upsilon) = \gcd\left(L_{a_0}, \frac{1}{L_{a_0}}(F_mG_{n+1} - G_nF_{m+1})\right) = L_{a_1} $$

- ${}_2\Upsilon = \Upsilon_{m+n-a_1+2} = \beta_{m-a_1+1}G_{n+1} - \alpha_{n-a_1+1}F_{m+1} + \frac{L_{a_0}}{L_{a_1}}\frac{F_m}{L_{a_0}}G_{n+2} - \frac{L_{a_0}}{L_{a_1}}\frac{G_n}{L_{a_0}}F_{m+2} =$
  . $= \beta_{m-a_1+1}G_{n+1} - \alpha_{n-a_1+1}F_{m+1} + \frac{F_m}{L_{a_1}}G_{n+2} - \frac{G_n}{L_{a_1}}F_{m+2},$
  where the pair α_{n-a_1+1} and β_{m-a_1+1} is a solution of the holynomial equation
  $\alpha_{n-a_1+1}\frac{F_m}{L_{a_0}} - \beta_{m-a_1+1}\frac{G_n}{L_{a_0}} = \frac{1}{L_{a_0}}\frac{1}{L_{a_1}}(F_mG_{n+1} - G_nF_{m+1})$

$$ {}_2L = \gcd({}_1L, {}_2\Upsilon) = \gcd\left({}_1L, \beta_{m-a_1+1}G_{n+1} - \alpha_{n-a_1+1}F_{m+1} + \frac{1}{L_{a_1}}(F_mG_{n+2} - G_nF_{m+2})\right) = L_{a_2} $$