21.07.2023 Branch of a curve F with the tangent y, where $y \nmid F_{m+1}$

Let y a tangent of the curve $F = F_m + F_{m+1} + \cdots$ of multiplicity k. Let $y \nmid F_{m+1}$. We denote $F_{m+j} = \sum f_i^0 x^{m+j-i} y^i$, therefore

$$
\begin{aligned}
F &= F_m + F_{m+1} + F_{m+2} + \cdots = \\
&= 0 + \cdots + 0 + f_k^0 x^{m-k} y^k + f_{k+1}^0 x^{m-k-1} y^{k+1} + \cdots + f_m^0 y^m + \\
&\quad + f_0^1 x^{m+1} + f_1^1 x^m y + \cdots + f_{m+1}^1 y^{m+1} + \\
&\quad + f_0^2 x^{m+2} + f^2 1_1 x^{m+1} y + \cdots + f_{m+2}^2 y^{m+2} + \\
&\quad + \cdots ,
\end{aligned}
$$

where $f_k^0 \neq 0$ and $f_0^1 \neq 0$. (*The coefficients use both upper and lower indices, therefore if we want to use their powers we put them in parentheses*) Then there is only one branch of F with the tangent y and its parametrization is of the form

$$
b(t) = \left(t^k, t^k \left(\sqrt[k]{\frac{-f_0^1}{f_k^0}} t + \gamma t^H + \cdots \right) \right) \tag{2}
$$

The first coefficient is proven in the dissertation thesis. After substituting the first partial parametrization

$$
\begin{aligned}
x &= x_1^k \\
y &= x_1^{k+1} \left(\sqrt[k]{\frac{-f_0^1}{f_k^0}} + y_1 \right)
\end{aligned}
$$

into F we get $F = x_1^{km+k} [F_1(x_1, y_1)]$, where

$$
\begin{aligned}
F_1 = \quad & y_1 \left(-k f_0^1 \left(\frac{f_k^0}{-f_0^1} \right)^{\frac{1}{k}} \right) + \\
& + x_1 \left(f_{k+1}^0 \left(\frac{-f_0^1}{f_k^0} \right)^{1+\frac{1}{k}} + f_1^1 \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{1}{k}} \right) + \\
& + x_1^2 \left(f_{k+2}^0 \left(\frac{-f_0^1}{f_k^0} \right)^{1+\frac{2}{k}} + f_2^1 \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{2}{k}} \right) + \\
& + x_1^3 \left(f_{k+3}^0 \left(\frac{-f_0^1}{f_k^0} \right)^{1+\frac{3}{k}} + f_3^1 \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{3}{k}} \right) + \\
& + \cdots \\
& + x_1^k \left(f_{2k}^0 \left(\frac{-f_0^1}{f_k^0} \right)^2 + f_k^1 \left(\frac{-f_0^1}{f_k^0} \right)^1 + f_0^2 \left(\frac{-f_0^1}{f_k^0} \right)^0 \right) + \\
& + x_1^{k+1} \left(f_{2k+1}^0 \left(\frac{-f_0^1}{f_k^0} \right)^{2+\frac{1}{k}} + f_{k+1}^1 \left(\frac{-f_0^1}{f_k^0} \right)^{1+\frac{1}{k}} + f_1^2 \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{1}{k}} \right) + \\
& + x_1^{k+2} \left(f_{2k+2}^0 \left(\frac{-f_0^1}{f_k^0} \right)^{2+\frac{2}{k}} + f_{k+2}^1 \left(\frac{-f_0^1}{f_k^0} \right)^{1+\frac{2}{k}} + f_2^2 \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{2}{k}} \right) + \\
& + \cdots \\
& + x_1^{2k} \left(f_{3k}^0 \left(\frac{-f_0^1}{f_k^0} \right)^3 + f_{2k}^1 \left(\frac{-f_0^1}{f_k^0} \right)^2 + f_k^2 \left(\frac{-f_0^1}{f_k^0} \right)^k + f_0^3 \left(\frac{-f_0^1}{f_k^0} \right)^0 \right) + \\
& + \cdots + \\
& + \text{(terms of higher or mixed degree)}
\end{aligned}
$$

The coefficient of the general term x_1^{pk+i} is

$$
\begin{aligned}
& x_1^{pk+i} \left(f_{(p+1)k+i}^0 \left(\frac{-f_0^1}{f_k^0} \right)^{p+1+\frac{i}{k}} + f_{pk+i}^1 \left(\frac{-f_0^1}{f_k^0} \right)^{p+\frac{i}{k}} + \cdots + f_i^{p+1} \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{i}{k}} \right) = \\
& x_1^{pk+i} \left(\sum_{j=0}^{p+1} f_{(p+1-i)k+i}^j \left(\frac{-f_0^1}{f_k^0} \right)^{p+1-i+\frac{i}{k}} \right)
\end{aligned}
$$

A little simpler form:

$$
\begin{aligned}
F_1 = \quad & y_1 \left(-k f_0^1 \left(\frac{f_k^0}{-f_0^1} \right)^{\frac{1}{k}} \right) + \\
& + x_1 \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{1}{k}} \left(f_{k+1}^0 \left(\frac{-f_0^1}{f_k^0} \right)^1 + f_1^1 \right) + \\
& + x_1^2 \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{2}{k}} \left(f_{k+2}^0 \left(\frac{-f_0^1}{f_k^0} \right)^1 + f_2^1 \right) + \\
& + x_1^3 \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{3}{k}} \left(f_{k+3}^0 \left(\frac{-f_0^1}{f_k^0} \right)^1 + f_3^1 \right) + \\
& + \cdots \\
& + x_1^k \quad \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{0}{k}} \left(f_{2k}^0 \left(\frac{-f_0^1}{f_k^0} \right)^2 + f_k^1 \left(\frac{-f_0^1}{f_k^0} \right)^1 + f_0^2 \right) + \\
& + x_1^{k+1} \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{1}{k}} \left(f_{2k+1}^0 \left(\frac{-f_0^1}{f_k^0} \right)^2 + f_{k+1}^1 \left(\frac{-f_0^1}{f_k^0} \right)^1 + f_1^2 \right) + \\
& + x_1^{k+2} \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{2}{k}} \left(f_{2k+2}^0 \left(\frac{-f_0^1}{f_k^0} \right)^2 + f_{k+2}^1 \left(\frac{-f_0^1}{f_k^0} \right)^1 + f_2^2 \right) + \\
& + \cdots \\
& + x_1^{2k} \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{0}{k}} \left(f_{3k}^0 \left(\frac{-f_0^1}{f_k^0} \right)^3 + f_{2k}^1 \left(\frac{-f_0^1}{f_k^0} \right)^2 + f_k^2 \left(\frac{-f_0^1}{f_k^0} \right)^k + f_0^3 \right) + \\
& + \cdots + \\
& + (\text{terms of higher or mixed degree})
\end{aligned}
$$

The coefficient of the general term x_1^{pk+i} is

$$
\begin{aligned}
& x_1^{pk+i} \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{i}{k}} \left(f_{(p+1)k+i}^0 \left(\frac{-f_0^1}{f_k^0} \right)^{p+1} + f_{pk+i}^1 \left(\frac{-f_0^1}{f_k^0} \right)^p + \cdots + f_{k+i}^p \left(\frac{-f_0^1}{f_k^0} \right)^1 + f_i^{p+1} \right) = \\
& x_1^{pk+i} \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{i}{k}} \left(\sum_{j=0}^{p+1} f_{(p+1-j)k+i}^j \left(\frac{-f_0^1}{f_k^0} \right)^{p+1-i} \right)
\end{aligned}
$$

We define a map

$$
V(p,i) = \sum_{j=0}^{p+1} f_{(p+1-j)k+j}^j (-f_0^1)^{p+1-j} (f_k^0)^j .
$$

Then we can calculate H and γ from the parametrization 2. $H = pk + i$ $(p, i \geq 0,\ i < k)$ is the number such that

$$
\begin{aligned}
V(0,1) = V(0,2) = \cdots = V(p, i-2) = 0 \\
V(p, i-1) \neq 0
\end{aligned}
$$

Then

$$
\gamma = \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{i+1}{k}} \frac{1}{k f_0^1 \, (f_k^0)^{p+1}} V(p,i)
$$

Príklad 1. *For example, for some k:*

- $H = 2 (= 0k + 2) \iff f_{k+1}^0 f_0^1 - f_k^0 f_1^1 \neq 0$. *Then*

$$
\gamma = \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{2}{k}} \frac{1}{k f_0^1 f_k^0} \left(f_{k+1}^0 f_0^1 - f_k^0 f_1^1 \right)
$$

- $H = 3 (= 0k + 3) \iff f_{k+1}^0 f_0^1 - f_k^0 f_1^1 = 0$ *and* $f_{k+2}^0 f_0^1 - f_k^0 f_2^1 \neq 0$. *Then*

$$
\gamma = \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{3}{k}} \frac{1}{k f_0^1 f_k^0} \left(f_{k+2}^0 f_0^1 - f_k^0 f_2^1 \right)
$$

- $H = k + 1 (= k + 1) \iff f_{k+1}^0 f_0^1 - f_k^0 f_1^1 = \cdots = f_{2k-1}^0 f_0^1 - f_{k-1}^0 f_2^1 \neq 0$, *and* $f_{2k}^0 \left(f_0^1\right)^2 - f_k^1 f_0^1 f_k^0 + f_0^2 \left(f_k^0\right)^2 = 0$. *Then*

$$
\gamma = \left(\frac{-f_0^1}{f_k^0} \right)^{\frac{1}{k}} \frac{1}{k f_0^1 \left(f_k^0\right)^2} \left(f_{2k}^0 \left(f_0^1\right)^2 - f_k^1 f_0^1 f_k^0 + f_0^2 \left(f_k^0\right)^2 \right)
$$